Mathematics Department Stanford University
Math 205A Sample Final Examination

Unless otherwise indicated, you can use results covered in lecture, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

3 Hours

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Q.6	
Q.7	
Q.8	
Q.10	
Q.11	
Q.12	
T/50	

Name (Print Clearly): __________________________

I understand and accept the provisions of the honor code (Signed) __________________________

1. (4 points) Let X be an topological space. Define "Borel regular outer measure on X." If μ is a Borel regular outer measure on X and if $A \subset X$ has the property that $\sup_{C \text{ closed}, C \subset A} \mu(C) = \mu(A) < \infty$, prove that A is μ-measurable.

2. (4 points) If $E \subset [0,1]$ and $\lambda(E) + \lambda([0,1] \setminus E) = 1$, prove that E is λ-measurable. (Here λ is Lebesgue outer measure on $\mathbb{R}$.)

3. (5 points) Give the definition of absolutely continuous (AC) function $f : [0,1] \to \mathbb{R}$ and prove that (i) the product fg of two AC functions $f, g : [0,1] \to \mathbb{R}$ is AC, and (ii) $\int_0^1 f\, g' = fg\big|_0^1 - \int_0^1 g\, f\,'$.

4. (4 points) Prove that f AC on $[0,1] \Rightarrow f$ is BV on $[0,1]$.

5. (5 points) If μ is an outer measure on a space X, define μ-measurability (in the sense of Caratheodory) of a set $A \subset X$, and give the proof that A_1, A_2 μ-measurable $\Longrightarrow A_1 \cup A_2$ is μ-measurable.

6. (3 points) Using the dominated convergence theorem or otherwise, prove that

$$\lim_{n\to\infty} \int_0^1 e^{1/x}(1+n^2x)^{-1}\sin(ne^{-1/x})\,dx = 0.$$

7. (5 points) State the 5-times covering lemma for collections $\mathcal{B}$ of closed balls contained in a bounded subset of $\mathbb{R}^n$. Using the 5-times covering lemma or otherwise, prove that if μ is a Borel measure on $\mathbb{R}^n$ such that $\liminf_{\rho\downarrow 0} \rho^{-n}\mu(B_\rho(x)) < \infty$ for μ-a.e. $x \in \mathbb{R}^n$, then μ is absolutely continuous with respect to Lebesgue measure (i.e. E Borel, $\lambda(E) = 0 \Longrightarrow \mu(E) = 0$).

8. (3 points) Suppose $A \subset [0, 1]$ is dense (thus $\overline{A} = [0, 1]$) and assume $f : A \to \mathbb{R}$ has the property that $\sum_{j=1}^{N} |f(x_j) - f(x_{j-1})| \le 1$ whenever $N \ge 1$ and $0 \le x_0 < x_1 < \cdots < x_N \le 1$ are points of A. Prove that there is a BV function g on $[0, 1]$ which agrees with f at each point of A.

Hint: Define $g(x) = f(x)$ if $x \in A$ and $g(x) = \limsup_{y\to x,\, y\in A} f(y)$ if $x \notin A$.

9. (5 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Prove that there is a Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = g(x)$ for Lebesgue a.e. $x \in \mathbb{R}^n$.

Hint: First consider the case when f is a non-negative simple function.

10. (5 points) Let $\mathcal{N} = \{1, 2, \ldots\}$, let $\mathcal{A}$ be the collection of all subsets of $\mathcal{N}$ and let μ be the counting measure on $\mathcal{N}$ (i.e. $\mu(A) =$ number of elements in the set A, taken to be 0 if $A = \emptyset$ and ∞ if A is an infinite subset.) In terms of series terminology, taking $a_n = f(n)$ do the following: (i) Assuming $f : \mathcal{N} \to [0, \infty]$, find the series expression for the value of $\int_{\mathcal{N}} f \, d\mu$ by directly applying the definition of the integral, (ii) Find, in series terminology, what it means for f to be μ integrable, (iii) Using only series terminology state the monotone convergence theorem and the dominated convergence theorem in this setting, and give the proof of each using a direct argument without reference to measure theory.

11. (4 points) Suppose $(X, \mathcal{A}, \mu)$ is any σ-finite measure space, $1 < p < \infty$, and $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable such that there is a constant $C > 0$ with $\int_X |fg|\, d\mu \leq C\|f\|_p$ for each $f \in \mathcal{L}^p(\mu)$. Prove that $g \in \mathcal{L}^q(\mu)$, where q is the conjugate exponent to p (i.e. $1/p + 1/q = 1$).

12. (3 points) Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be arbitrary measure spaces and let γ be the product outer measure defined as usual by $\gamma(A) = \inf \sum_i \mu(A_i)\nu(B_i)$ with the inf over all collections $\{A_i \times B_i\}_{i=1,2,...}$ with $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$, and $A \subset \cup_i A_i \times B_i$.

Prove that $\gamma(\cup_j E_j \times F_j) = \sum_j \mu(E_j)\nu(F_j)$ whenever $E_j \in \mathcal{A}$, $F_j \in \mathcal{B}$ and the sets $E_1 \times F_1, E_2 \times F_2, \ldots$ are pairwise disjoint.

Note: Your proof should not depend on Fubini's theorem—recall that the above lemma was proved as part of the preliminary discussion needed in the eventual proof of Fubini's theorem.