Stanford Mathematics Department
Math 205A Lecture Supplement #5
Riesz Representation for $L^p(\mu)$

Here $(X, \mathcal{A}, \mu)$ is any measure space and $1 \le p \le \infty$, $1 \le q \le \infty$ are "conjugate exponents," meaning that

$$(*) \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

where of course we take $\frac{1}{\infty} = 0$. $\mathcal{L}^p(\mu)$ will here, for $1 \le p < \infty$, denote the real-valued $\mathcal{A}$-measurable functions f such that $\int_X |f|^p \, d\mu < \infty$, equipped with the seminorm

$$\|f\|_p = \Big(\int_X |f|^p \, d\mu\Big)^{1/p},$$

and $\mathcal{L}^\infty(\mu)$ denotes the set of μ-essentially bounded real-valued functions f (i.e. the $\mathcal{A}$-measurable functions $f : X \to \mathbb{R}$ such that there is $\lambda \in (0, \infty)$ with $|f| \le \lambda$ μ-a.e.) and we let

$$\|f\|_\infty = \inf\{\lambda \in (0, \infty) : |f(x)| \le \lambda \text{ for } \mu\text{-a.e. } x \in X\}.$$

In this section we discuss the dual space $(L^p(\mu))^*$ of $L^p(\mu)$. Thus $(L^p(\mu))^*$ is the set of bounded linear functionals F on $L^p(\mu)$, so $F \in (L^p(\mu))^*$ means that $F : L^p(\mu) \to \mathbb{R}$ is a linear map with $\|F\| = \sup_{\|f\|_p \le 1} |F(f)| < \infty$.

To begin, recall the Hölder inequality

$$\int_X |fg| \, d\mu \le \|f\|_p \|g\|_q < \infty, \quad f \in \mathcal{L}^p(\mu),\ g \in \mathcal{L}^q(\mu),$$

so if we define

$$T_g(\tilde{f}) = \int_X fg \, d\mu, \quad f \in L^p(\mu),$$

where $\tilde{f}$ denotes the L^p class of $f \in \mathcal{L}^p(\mu)$ $(= \{h : h = f \text{ a.e. in } X\})$, then T_g is a bounded linear map of $L^p(\mu)$ into $\mathbb{R}$; that is $g \in L^q(\mu) \Rightarrow T_g \in (L^p(\mu))^*$. Notice that we also have linearity in g; that is if $g_1, g_2 \in L^q(\mu)$ and $\lambda, \eta \in \mathbb{R}$ then $T_{c_1 g_1 + c_2 g_2} = c_1 T_{g_1} + c_2 T_{g_2}$. Thus map

$$(**) \qquad T : g \mapsto T_g$$

defines a linear map $L^q(\mu) \to (L^p(\mu))^*$. The following Riesz theorem claims that T, so defined, is an isometric isomorphism of $L^q(\mu)$ onto $(L^p(\mu))^*$ provided that in the case $p = 1$ we make the additional assumption that μ is σ-finite.

Theorem (Riesz Representation for L^p.) *Let $1 \le p < \infty$, and let $(X, \mathcal{A}, \mu)$ be any measure space for $p \ne 1$ and $(x, \mathcal{A}, \mu)$ be any σ-finite measure space in the case $p = 1$, and let q be the exponent conjugate to p as in $(*)$. Then the map T in $(**)$ is an isometric isomorphism of $L^q(\mu)$ onto the dual space $(L^p(\mu))^*$ of $L^p(\mu)$.*

Proof: It was shown in Q.6 of hw8 that, under the conditions stated in the above theorem, T defined as in $(**)$ is an isometry of $L^q(\mu)$ into $(L^p(\mu))^*$ (i.e. that $\|T_g\| = \|g\|_q$ where $\|T_g\| = \sup_{\|f\|_p \le 1} |T_g(f)|$).

Thus we merely have to prove that T is onto. That is for any given bounded linear functional $F : L^p(\mu) \to \mathbb{R}$ we have to prove there is a $g \in L^q(\mu)$ with $F = T_g$. So assume a linear $F : L^p(\mu) \to \mathbb{R}$ is given with $\|F\| < \infty$, where as usual $\|F\| = \sup_{\|f\|_p = 1} |F(f)|$. We consider cases, beginning with:

Case 1: $\mu(X) < \infty$. In this case we define $\nu : \mathcal{A} \to \mathbb{R}$ by

$$\nu(A) = F(\widetilde{\chi_A}),$$

where χ_A denotes the indicator function of A and $\tilde{f}$ as usual denotes the $L^p(\mu)$ class of a function $f \in \mathcal{L}^p(\mu)$. We claim that ν is a signed measure. To check this, first observe that $\widetilde{\chi_\emptyset} = 0$, the zero class in $L^p(\mu)$, and hence $F(\widetilde{\chi_\emptyset}) = 0$, so $\nu(\emptyset) = 0$. Also, if $A_1, A_2, \ldots$ are p.w.d. sets in $\mathcal{A}$ then $\nu(\cup_{j=1}^N A_j) = F(\widetilde{\chi_{\cup_{j=1}^N A_j}}) = \sum_{j=1}^N F(\widetilde{\chi_{A_j}})$ and taking limits as $N \to \infty$ we see that $\nu(\cup_{j=1}^N A_j)$ converges to both $\sum_{j=1}^\infty F(\widetilde{\chi_{A_j}})$ and $F(\widetilde{\chi_{\cup_{j=1}^\infty A_j}})$, so $\nu(\cup_{j=1}^\infty A_j) = \sum_{j=1}^\infty \nu(A_j)$, and hence indeed ν is a signed measure. Furthermore it is finite (i.e. $|\nu(A)| < \infty$ for each $A \in \mathcal{A}$) and the argument above to prove $\nu(\emptyset) = 0$ actually shows that $\nu(E) = 0$ whenever $E \in \mathcal{A}$ with $\mu(E) = 0$, because the indicator function χ_E of any set of measure zero is in the L^p class of the zero function. Thus $E \in \mathcal{A}$ with $\mu(E) = 0 \Rightarrow \nu(E) = 0$. Thus if we let $P, X \setminus P$ be a Hahn decomposition for ν then $\nu = \nu \llcorner P + \nu \llcorner (X \setminus P)$ and both $\nu_1 = \nu \llcorner P$ and $\nu_2 = -\nu \llcorner (X \setminus P)$ are positive measures on $\mathcal{A}$ which are AC with respect to μ, hence by the Radon-Nikodym Theorem there are $\mathcal{A}$ measurable functions $g_1, g_2 : X \to [0, \infty)$ with $\nu_j(A) = \int_A g_j \, d\mu$, $j = 1, 2$, hence

$$(1) \qquad \nu(A) = F(\widetilde{\chi_A}) = \int_A g \, d\mu, \quad A \in \mathcal{A}, \quad g = g_1 - g_2.$$

By the linearity of each side of (1) we then have

$$(2) \qquad \int_X \varphi g \, d\mu = F(\tilde{\varphi}), \quad \text{for any simple function } \varphi.$$

Next notice that if $f \in \mathcal{L}^p(\mu)$ then by a theorem of lecture we can find increasing sequences ψ_i, η_i of non-negative simple functions with $\psi_i \to f_+(= \max\{f, 0\})$ and $\eta_i \to f_-(= \max\{-f, 0\})$ pointwise on all of X and hence $0 \le (f_+ - \psi_i)^p \to 0$ and $0 \le (f_+ - \psi_i)^p \le f_+^p$ so by the Dominated Convergence Theorem $\|f_+ - \psi_i\|_p \to 0$, and similarly $\|f_- - \eta_i\|_p \to 0$. Hence we have shown (with $\varphi_i = \psi_i - \eta_i$)

$$(3) \qquad f \in \mathcal{L}^p(\mu) \Rightarrow \exists \text{ simple functions } \varphi_i \text{ with } \|f - \varphi_i\|_p \to 0.$$

If $1 < p < \infty$, we apply (3) to $f = f_k$, where

$$f_k = (\operatorname{sgn} g)|g|^{q/p}\chi_{G_k}, \quad \text{where } G_k = \{x \in X : |g(x)| < k\}.$$

In that case we can set $\varphi = \varphi_i$ on each side of (2) where $\|f_k - \varphi_i\|_p \to 0$ and hence by taking the limit of each side as $i \to \infty$ we obtain

$$F(\widetilde{f_k}) = \int_X f_k g = \int_{G_k} |g|^{1+q/p}\, d\mu, \quad k = 1, 2, \ldots.$$

But $F(\widetilde{f_k}) \le \|F\| \|f_k\|_p = \|F\| (\int_{G_k} |g|^q\, d\mu)^{1/p}$ and hence

$$\int_{G_k} |g|^{1+q/p}\, d\mu \le \|F\| (\int_{G_k} |g|^q\, d\mu)^{1/p},$$

hence, since $1 + q/p = q$,

$$\|g\chi_{G_k}\|_q \le \|F\|.$$

Letting $k \to \infty$ and using the Monotone Convergence Theorem we thus have $g \in L^q(\mu)$. In the case when $p = 1, q = \infty$ the argument is similar except that we use $f_k = (\operatorname{sgn} g)|g|^Q\chi_{G_k}$, where again $G_k = \{x \in X : |g(x)| < k\}$, where $Q > 0$ is arbitrary. Then using (2) as in the case $p > 1$ we get this time that

$$\int_{G_k} |g|^{1+Q}\, d\mu \le \|F\| \int_{G_k} |g|^Q\, d\mu$$

and by using Hölder to give $\int_{G_k} |g|^Q\, d\mu \le (\int_{G_k} |g|^{1+Q}\, d\mu)^{Q/(1+Q)}(\mu(G_k)^{1/(1+Q)})$ we obtain

$$\Big(\int_{G_k} |g|^{1+Q}\, d\mu\Big)^{1/(1+Q)} \le \|F\|\, \mu(X)^{1/(1+Q)}$$

and hence by letting $Q \to \infty$ we get (see Q.2 of hw7)

$$\|g\chi_{G_k}\|_\infty \le \|F\|, \quad k = 1, 2, \ldots,$$

and hence $\|g\|_\infty < \infty$. Thus in either case $p = 1, p > 1$ we have proved $g \in \mathcal{L}^q(\mu)$, and for any $f \in \mathcal{L}^p(\mu)$ we can let $\varphi = \varphi_i$ in (2) and use (3) to pass to the limit, giving

$$\int_X fg\, d\mu = F(\widetilde{f}),$$

so indeed (in both cases $p = 1$, $p > 1$) we have $F(\widetilde{f}) = T_g(f)$. This completes the proof in the case $\mu(X) < \infty$.

Case 2: μ is σ-finite. Thus we assume $\mu(X) = \infty$ and that there are p.w.d. sets $B_1, B_2, \ldots \in \mathcal{A}$ with $\mu(B_j) < \infty$. Then we can apply Case 1 to the measure space $(B_j, \mathcal{A}_j, \mu_j)$, where $\mathcal{A}_j = \{A \cap B_j : A \in \mathcal{A}\}$ and $\mu_j = \mu|\mathcal{A}_j$ and with F_j in place of F, where $F_j(\widetilde{f}) = F(\widetilde{f_j})$ for $f \in \mathcal{L}^p(\mu_j)$, where f_j the $\mathcal{L}^p(\mu)$ function defined $f_j|B_j = f$ and $f_j|X \setminus B_j = 0$. Thus there is $g_j^0 \in \mathcal{L}^q(\mu_j)$ with $\int_X f_j g_j\, d\mu = F(f_j)$, where $g_j|B_j = g_j^0$ and $g_j|X \setminus B_j = 0$. Thus

$$\int_X fg_j = F(\widetilde{\chi_{B_j} f}), \quad f \in \mathcal{L}^p(\mu),\ j = 1, 2, \ldots.$$

Since the B_j are p.w.d. this can be written

$$\int_X f\chi_{B_j} g = F(\widetilde{\chi_{B_j} f}), \quad f \in \mathcal{L}^p(\mu),\ j = 1, 2, \ldots,$$

where $g|B_j = g_j$ for each j and $g|X \setminus (\cup_{j=1}^\infty B_j) = 0$, and by linearity this in turn gives

$$(*) \qquad \int_X f\chi_{\cup_{j=1}^N B_j} g = F(\widetilde{\chi_{\cup_{j=1}^N B_j} f}), \quad f \in \mathcal{L}^p(\mu),\ N = 1, 2, \ldots,$$

and (Cf. the argument used in Case 1) we then have

$$\|g\chi_{\cup_{j=1}^N B_j}\|_q \le \|F\|, \quad N = 1, 2, \ldots,$$

and for $q < \infty$ we can apply the monotone convergence theorem on the left to give

$$\|g\|_q \le \|F\| < \infty.$$

Of course the same is trivially true in the case $q = \infty$ because $\cup_{j=1}^\infty B_j = X$ and hence $\|g\chi_{\cup_{j=1}^N B_j}\|_\infty \to \|g\|_\infty$. We can then let $N \to \infty$ in $(*)$ to conclude $F(f) = \int_X fg\, d\mu$, so the proof is complete in Case 2.

Thus it remains to treat Case 3, the case when $1 < p < \infty$, $\mu(X) = \infty$, and when no σ-finite hypothesis is assumed. To give the proof in this case we let

$$\mathcal{E} = \{E \in \mathcal{A} : E = \cup_{j=1}^\infty E_j \text{ for some } E_j \in \mathcal{A} \text{ with } \mu(E_j) < \infty\ \forall j\}.$$

Then for each $E \in \mathcal{E}$ we can apply Case 2 above to the measure space $(E, \mathcal{A}_E, \mu_E)$, where $\mathcal{A}_E = \{A \cap E : A \in \mathcal{A}\}$ and $\mu_E(A) = \mu(A \cap E)$ for each $A \in \mathcal{A}$, to give a $g_E^0 \in \mathcal{L}^q(\mu_E)$ such that

$$\int_E fg_E^0\, d\mu_E = F_E(\widetilde{f}), \quad f \in L^p(\mu_E),$$

where $F_E(\tilde{f}) = F(\tilde{f_E})$, with $f_E \in \mathcal{L}^p(\mu)$ defined by $f_E|E = f$ on E and $f_E|X \setminus E = 0$. Thus in fact

$$(\ddagger) \qquad \int_X f g_E \, d\mu = F(\widetilde{\chi_E f}), \quad f \in L^p(\mu),\ E \in \mathcal{E},$$

where we use the notation $g_E = g_E^0$ on E and $g_E = 0$ on $X \setminus E$ for each $E \in \mathcal{E}$. Then as in Case 2 we have $\|g_E\|_q \le \|F\|$ for each $E \in \mathcal{E}$, so

$$\alpha = \sup_{E \in \mathcal{E}} \|g_E\|_q < \infty$$

and we can choose a sequence $E_1, E_2, \ldots \in \mathcal{E}$ with $\|g_{E_j}\|_q \to \alpha$.

Now observe that $E, H \in \mathcal{E}$ with $E \subset H \Rightarrow g_H = g_E$ a.e. in E which is easily checked because $(\ddagger)$ implies that $\int_E f(g_H - g_E)\, d\mu = 0$ for each $f \in \mathcal{L}^p(\mu)$, so we can choose $f = \operatorname{sgn}(g_H - g_E)|g_H - g_E|^{q/p}\chi_E$ (which is an $\mathcal{L}^p(\mu)$ function), and hence (since $1 + q/p = q$)

$$\int_E |g_H - g_E|^q = 0.$$

Thus

$$E, H \in \mathcal{E} \text{ with } E \subset H \Rightarrow \|g_E\|_q \le \|g_H\|_q,$$

with equality if and only if $g_H = 0$ a.e. on $X \setminus E$. In particular $\|g_{E_j}\|_q \to \alpha$ implies $\|g_{\cup_{j=1}^\infty E_j}\|_q = \alpha$ and also $H \in \mathcal{E}$ with $H \supset \cup_{j=1}^\infty E_j \Rightarrow g_H = 0$ a.e. on $X \setminus (\cup_{j=1}^\infty E_j)$, otherwise we contradict the definition of α. Since $f \in \mathcal{L}^p(\mu)$ evidently implies $H_f = \{x \in X : |f(x)| \ne 0\} \cup (\cup_{j=1}^\infty E_j)$ is in the collection $\mathcal{E}$, we must then in particular have $g_{H_f} = 0$ a.e. on $X \setminus (\cup_{j=1}^\infty E_j)$ and so, with $g = g_{\cup_{j=1}^\infty E_j}$,

$$F(\tilde{f}) = \int_X fg \, d\mu \ \ \forall f \in \mathcal{L}^p(\mu),$$

and the proof is complete. □