Mathematics Department Stanford University
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Product measures and Fubini's theorem

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be arbitrary measure spaces.

Definition: By an $\mathcal{A},\mathcal{B}$-rectangle we mean any set of the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

The product outer measure γ on $X \times Y$ corresponding to the two given measure spaces is defined as follows. For any set $S \subset X \times Y$,

$$\gamma(S) = \inf \sum_i \mu(A_i)\nu(B_i),$$

where the inf is taken over all countable collections $\{A_i \times B_i\}$ of $\mathcal{A},\mathcal{B}$-rectangles such that $S \subset \cup_i A_i \times B_i$. It is left as an exercise to check that γ is indeed an outer measure on $X \times Y$, and where the usual convention that $0.\infty = \infty.0 = 0$ is adopted.

We aim to prove that the σ-algebra of γ-measurable sets (in the sense of Caratheodory) contains all the $\mathcal{A},\mathcal{B}$-rectangles. The first non-trivial thing to check is the following countable additivity property:

Lemma 1. *If $A_1 \times B_1, A_2 \times B_2, \ldots$ are pairwise-disjoint $\mathcal{A},\mathcal{B}$-rectangles, then*

$$\gamma(\cup_i A_i \times B_i) = \sum_i \mu(A_i)\nu(B_i).$$

Proof: Notice that the inequality $\gamma(A_i \times B_i) \le \mu(A_i)\nu(B_i)\ \forall i$ is trivial by the definition of γ, so by the subadditivity of outer measure we have $\gamma(\cup_i A_i \times B_i) \le \sum_i \mu(A_i)\nu(B_i)$ and we have only to prove the reverse inequality. So let $\{C_i \times D_i\}$ be any countable collection with $\cup_j A_j \times B_j \subset \cup_i C_i \times D_i$, and notice that then

$$\sum_i \chi_{A_i}(x)\chi_{B_i}(y) \equiv \sum_i \chi_{A_i \times B_i}(x,y) \le \sum_i \chi_{C_i \times D_i} \equiv \sum_i \chi_{C_i}(x)\chi_{D_i}(y).$$

Taking fixed $x \in X$, and integrating with respect to $y \in Y$ we then deduce that

$$\sum_i \chi_{A_i}(x)\nu(B_i) \le \sum_i \chi_{C_i}(x)\nu(D_i),$$

whence integrating with respect to $x \in X$ we conclude

$$\sum_i \mu(A_i)\nu(B_i) \le \sum_i \mu(C_i)\nu(D_i),$$

and by taking the inf over all such collections $\{C_i \times D_i\}$ we then conclude by definition of γ that

$$\sum_i \mu(A_i)\nu(B_i) \le \gamma(\cup_i A_i \times B_i)$$

as required.

Next we have the fact that $\mathcal{A},\mathcal{B}$-rectangles are γ-measurable:

Lemma 2. *Any $\mathcal{A},\mathcal{B}$-rectangle $A \times B$ is γ-measurable in the sense of Caratheodory.*

Before we begin the proof, we need the facts in the following remarks:

Remarks: (1) A countable (or finite) intersection of $\mathcal{A},\mathcal{B}$-rectangles is again an $\mathcal{A},\mathcal{B}$-rectangle, and if $S_1, \ldots, S_j$, T are $\mathcal{A},\mathcal{B}$-rectangles, then $T \setminus \cup_{i=1}^j S_j$ is a union of a finite collection of pairwise disjoint $\mathcal{A},\mathcal{B}$-rectangles, as one easily checks by induction on j. (Check: For $j = 1$ it is true because $A \times B \setminus C \times D$ can be written as the disjoint union of the $\mathcal{A},\mathcal{B}$ rectangles $(A \cap C) \times (B \setminus D)$, $(A \setminus C) \times B$, while for $j \ge 2$ we can write $T \setminus \cup_{i=1}^j S_i = (T \setminus \cup_{i=1}^{j-1} S_i) \setminus S_j$, and we can apply the case $j = 1$ and induction on j to show that this is indeed a disjoint union of $\mathcal{A},\mathcal{B}$-rectangles.)

(2) Notice that it follows from (1) that if $\{A_i \times B_i\}$ are given $\mathcal{A},\mathcal{B}$-rectangles for $i = 1, 2, \ldots$, then $\cup_i A_i \times B_i$ can be written as the pairwise-disjoint union $\cup_i C_i \times D_i$ of $\mathcal{A},\mathcal{B}$-rectangles, because $\cup_{i=1}^{\infty} A_i \times B_i = \cup_{i=1}^{\infty}(A_i \times B_i \setminus (\cup_{j=0}^{i-1} A_j \times B_j))$, where we use the notation that $A_0 = B_0 = \emptyset$.

Proof of Lemma 2: Let $Z \subset X \times Y$ be arbitrary, and let $A_i \times B_i$ be $\mathcal{A},\mathcal{B}$-rectangles with $Z \subset \cup_i A_i \times B_i$. Then by monotonicity and subadditivity of the outer measure γ we have $\gamma(Z \cap (A \times B) + \gamma(Z \setminus (A \times B)) \le \gamma((\cup_i A_i \times B_i) \cap (A \times B)) + \gamma((\cup_i A_i \times B_i) \setminus (A \times B)) \le \sum_i (\gamma((A_i \times B_i) \cap (A \times B)) + \gamma((A_i \times B_i) \setminus (A \times B)))$, and by Remarks 1 and 2 above we have $(A_i \times B_i) \cap (A \times B)) \cup ((A_i \times B_i) \setminus (A \times B))$ is a pairwise pairwise disjoint union of 3 $\mathcal{A},\mathcal{B}$-rectangles, and the union is equal to $A_i \times B_i$, so by Lemma 1 $\gamma((A_i \times B_i) \cap (A \times B)) + \gamma((A_i \times B_i) \setminus (A \times B)) = \mu(A_i)\nu(B_i)$ for each i. Thus we have shown $\gamma(Z \cap (A \times B) + \gamma(Z \setminus (A \times B)) \le \sum_i \mu(A_i)\nu(B_i)$, and by taking inf over all such collections $\{A_i \times B_i\}$ we have $\gamma(Z \cap (A \times B) + \gamma(Z \setminus (A \times B)) \le \gamma(Z)$ as required.

In view of the fact that the sets which are measurable with respect to a given outer measure form a σ-algebra, we thus have:

Corollary. *The collection of γ-measurable sets contains the σ-algebra generated by all the $\mathcal{A},\mathcal{B}$-rectangles.*

Remark 3: Observe that now we can check that if $(X, \mathcal{A}, \mu) = (\mathbb{R}^{n-1}, \mathcal{M}_{n-1}, \lambda_{n-1})$ and $(Y, \mathcal{B}, \nu) = (\mathbb{R}, \mathcal{M}_1, \lambda_1)$ (where $\mathcal{M}_j$ denotes the Lebesgue measurable subsets of $\mathbb{R}^j$ and λ_j denotes the restriction to $\mathcal{M}_j$ of Lebesgue outer measure on $\mathbb{R}^j$), then γ is just Lebesgue outer measure on $\mathbb{R}^n$. Since we proved in lecture that for each $A \subset \mathbb{R}^j$ we can find a countable intersection $E = \cap_j U_j$ of open sets with $U_j \supset A$ for each j and $\lambda(E) = \lambda(A)$, it is straightforward then to check that γ is Borel regular in case $(X, \mathcal{A}, \mu) = (\mathbb{R}^{n-1}, \mathcal{M}_{n-1}, \lambda_{n-1})$ and $(Y, \mathcal{B}, \nu) = (\mathbb{R}, \mathcal{M}_1, \lambda_1)$. Of course all open sets are also γ-measurable in this case by Corollary 1, because any open set is a countable union of open intervals, which are $\mathcal{A},\mathcal{B}$ rectangles in the present setting. Thus in this case γ is a Borel regular outer measure on $\mathbb{R}^n$ with $\gamma(I) = |I|$ for each open interval $I \subset \mathbb{R}^n$ (by Lemma 1), and hence $\gamma = \lambda$ by virtue of Q.5 of Homework 4.

The following lemma provides the main ingredient in the proof of Fubini's theorem.

Lemma 3. *If $\{A_i \times B_i\}$ is any countable collection of $\mathcal{A},\mathcal{B}$ rectangles, then*

$$\gamma(\cup_i A_i \times B_i) = \int_{X \times Y} \chi_{\cup_i A_i \times B_i}\, d\gamma = \int_Y \Big(\int_X \chi_{\cup A_i \times B_i}(x, y)\, d\mu(x) \Big)\, d\nu(y)$$

(and all integrals are well-defined).

Proof: Indeed the γ-measurability of $\cup_i A_i \times B_i$ is guaranteed by Lemma 2, so the integral $\int_{X \times Y} \chi_{\cup_i A_i \times B_i}\, d\gamma$ is defined and is equal to $\gamma(\cup_i A_i \times B_i)$. But by Remark 1 above we can write $\cup_i A_i \times B_i = \cup_i C_i \times D_i$ where $C_i \times D_i$ are p.w.d. $\mathcal{A},\mathcal{B}$ rectangles. So by Lemma 1 $\gamma(\cup_i A_i \times B_i) = \sum_i \mu(C_i)\nu(D_i)$ which of course can be written as $\sum_i \int_Y \big(\int_X \chi_{C_i \times D_i}(x, y)\, d\mu(x)\big)\, d\nu(y)$, which by two applications of the monotone convergence theorem is the same as $\int_Y \big(\int_X \sum_i \chi_{C_i \times D_i}(x, y)\, d\mu(x)\big)\, d\nu(y)$, which is just $\int_Y \big(\int_X \chi_{\cup A_i \times B_i}(x, y)\, d\mu(x)\big)\, d\mu(y)$, so the identity of Lemma 3 is proved.

We can now state Fubini's Theorem. In the statement we require that the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be <u>complete</u>; a measure space $(X, \mathcal{A}, \mu)$ is said be complete if $E \in \mathcal{A}$, $\mu(E) = 0 \Rightarrow$ all subsets of E are also in $\mathcal{A}$. (Of course all such subsets must then trivially have μ-measure zero.) Observe also that such completeness trivially holds if $\mu = \mu_0 | \mathcal{A}$, where μ_0 is an outer measure on X and $\mathcal{A}$ is the collection of all subsets which are μ_0-measurable in the sense of Caratheodory. (Because all sets with μ_0-measure zero are trivially μ_0-measurable in the sense of Caratheodory.)

Remark 4: Observe that in a complete measure space $(X, \mathcal{A}, \mu)$ we have the very convenient fact that if $f, g : X \to [-\infty, \infty]$, if g is $\mathcal{A}$-measurable, and if $f = g$ μ-a.e. (i.e. there is a set $E \in \mathcal{A}$ of measure zero such that $f \equiv g$ on $X \setminus E$), then f is automatically $\mathcal{A}$-measurable. Because of this we can make perfectly good sense of integration of functions which are almost everywhere equal to an integrable function but which may not even be defined on some set of measure zero; in this case we simply arbitrarily define the function to be (for example) zero on the set of measure zero where it is not otherwise defined. We subsequently adopt this convention whenever we are in a complete measure space.

Theorem (Fubini's Theorem). *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete measure spaces, let γ be the product outer measure on $X \times Y$ constructed above, and suppose that $f : X \times Y \to \mathbb{R}$ is γ-integrable. Then*
(i) $f(x,y)$ is a μ-integrable function of x for ν-a.e. $y \in Y$;
(ii) $\int_X f(x,y)\, d\mu(x)$ is a ν-integrable function of y;
(iii) $\int_Y (\int_X f(x,y)\, d\mu(x))\, d\nu(y) = \int_{X\times Y} f(x,y)\, d\gamma$.

Remark 5: Notice that the integral in (ii) exists by virtue of conclusion (i) and the iterated integral on the left in (iii) exists by virtue of conclusion (ii); also (in accordance with Remark 4 above) it is understood in (ii), (iii) that we adopt the convention that $\int_X f(x,y)\, d\mu(x)$ is defined to be zero at the (ν-measure zero) set of points y where it is not otherwise defined.

Proof: We first show this is correct when $f = \chi_{\cap_j(\cup_i A_i^j \times B_i^j)}$, where each $A_i^j \times B_i^j$ is an $\mathcal{A}, \mathcal{B}$ rectangle and $\gamma(\cup_i A_i^1 \times B_i^1) < \infty$. Indeed, since for each $k = 1, 2, \ldots$ $\cap_{j=1}^k(\cup_i A_i^j \times B_i^j)$ is a countable union of $\mathcal{A}, \mathcal{B}$ rectangles, Lemma 3 tells us that

$$\int_{X\times Y} \chi_{\cap_{j=1}^k(\cup_i A_i^j \times B_i^j)}\, d\gamma = \int_Y \Big(\int_X \chi_{\cap_{j=1}^k(\cup A_i^j \times B_i^j)}(x,y)\, d\mu(x)\Big)\, d\nu(y)$$

and so we can make 3 applications of the dominated convergence theorem (once on the left side of (1) and twice on the right side) to conclude that

$$\text{(1)}\quad \gamma(\cap_{j=1}^\infty(\cup_i A_i^j \times B_i^j))) = \int_{X\times Y} \chi_{\cap_{j=1}^\infty(\cup_i A_i^j \times B_i^j)}\, d\gamma = \int_Y \Big(\int_X \chi_{\cap_{j=1}^\infty(\cup A_i^j \times B_i^j)}(x,y)\, d\mu(x)\Big)\, d\nu(y).$$

Next notice that using the definition of the outer measure γ, we additionally conclude the following: For every γ-measurable set C of finite measure, we can select, for each $j = 1, 2, \ldots$, p.w.d. families $\{A_i^j \times B_i^j : i = 1, 2, \ldots\}$ of $\mathcal{A}, \mathcal{B}$ rectangles with

$$C \subset \cap_j(\cup_i A_i^j \times B_i^j) \text{ and } \gamma(\cap_j(\cup_i A_i^j \times B_i^j \setminus C) = 0.$$

Then by applying the same reasoning with $E = \cap_j(\cup_i A_i^j \times B_i^j) \setminus C$ in place of C we also get families $\{E_i^j \times F_i^j : i = 1, 2, \ldots\}$ of $\mathcal{A}, \mathcal{B}$ rectangles with

$$E \subset (\cap_j(\cup_i E_i^j \times F_i^j))$$

and

$$0 = \gamma(\cap_j(\cup_i E_i^j \times F_i^j)) = \int_{X\times Y} \chi_{\cap_j(\cup_i E_i^j \times F_i^j)}\, d\gamma = \int_Y \Big(\int_X \chi_{\cap_j(\cup_i E_i^j \times F_i^j)}\, d\mu\Big)\, d\nu,$$

which implies that $\int_X \chi_{\cap_j(\cup_i E_i^j \times F_i^j)}(x,y)\, d\mu(x) = 0$ for $\nu-$a.e. $y \in Y$. That is the "y-slice" $\{x \in X : (x,y) \in \cap_j(\cup_i E_i^j \times F_i^j)\}$ (which is a set in $\mathcal{A}$) has ν-measure zero for ν-a.e. $y \in Y$. But $E \subset \cap(\cup_i E_i^j \times F_i^j))$ and ν is a complete measure, so the slice $\{x : (x,y) \in E\}$ is also in $\mathcal{A}$ and also has μ-measure zero for ν-a.e. $y \in E$. Thus $\int_Y \big(\int_X \chi_E(x,y)\, d\mu(x)\big) d\nu(y) = 0$ (and the integrals are well-defined), because ν is a complete measure and hence we can use the convention discussed in Remark 4 above. Thus we have

$$\text{(2)}\qquad \int_{X\times Y} \chi_E\, d\gamma = \int_Y \Big(\int_X \chi_E(x,y)\, d\mu(x)\Big) d\nu(y) = 0.$$

Since $E = \cap_j(\cup_i A_i^j \times B_i^j) \setminus C$ and $C \subset \cap_j(\cup_i A_i^j \times B_i^j)$, we then have $\chi_C = \chi_{\cap_j(\cup_i A_i^j \times B_i^j)} - \chi_E$ and in view of (1), (2) and the linearity of the integral we have

$$\int_{X\times Y} \chi_C \, d\gamma = \int_Y \Big(\int_X \chi_C(x,y)\, d\mu(x)\Big) d\nu(y) \tag{3}$$

(and all integrals are well-defined), provided C is γ-measurable and has finite measure.

We can now easily complete the proof because (3) plus the linearity of the integral implies

$$\int_{X\times Y} \varphi \, d\gamma = \int_Y \Big(\int_X \varphi(x,y)\, d\mu(x)\Big) d\nu(y) \tag{4}$$

for any simple function $\varphi = \sum_{j=1}^N c_j \chi_{C_j}$ with $c_j > 0$ for each $j = 1, \ldots, N$, provided the C_j are γ-measurable and $\gamma(C_j) < \infty$. So suppose without loss of generality (since we can write $f = f_+ - f_-$ and use the linearity of the integral) that $f \geq 0$ and select an increasing sequence of non-negative simple functions $\varphi_k = \sum_{j=1}^{N_k} c_j^k \chi_{C_j^k}$ with $c_j^k \geq 0$, each C_j^k is γ-measurable, and $\varphi_k \to f$. Observe that then $\gamma(C_j^k) < \infty$ for each j, k such that $c_j^k > 0$ because $\int_{X\times Y} \varphi_k \, d\gamma \leq \int_{X\times Y} f \, d\gamma < \infty$, so we can apply (4) with φ_k in place of φ. Then by applying the monotone convergence theorem (once on the left side, and twice on the right side) we conclude Fubini's Theorem as claimed.

If f is non-negative the hypothesis in Fubini's Theorem that f is integrable can be replaced by the weaker hypothesis that $f : X \times Y \to [0, \infty]$ is merely γ-measurable, provided that the given measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite. This is known as Tonelli's theorem:

Corollary (Tonelli's Theorem). *If the spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite complete measure spaces, and if $f : X \times Y \to [0, \infty]$ is γ-measurable, then*

$$\int_{X\times Y} f \, d\gamma = \int_Y \Big(\int_X f(x,y)\, d\mu(x)\Big) d\nu(y)$$

(and all integrals are well-defined).

Remark 6: Notice that here, unlike Fubini's Theorem, we allow the possibility that $\int_{X\times Y} f \, d\gamma = \infty$, so for a given γ-measurable function $f : X \times Y \to \mathbb{R}$, integrable or not, we can apply Tonelli's Theorem to $|f|$, enabling us to actually check whether f is integrable or not. If it is integrable then we can of course apply Fubini's theorem to evaluate the integral.

Proof of Tonelli's Theorem: Let $A_k \in \mathcal{A}$, $B_k \in \mathcal{B}$ be increasing sequences with $\mu(A_k) < \infty$, $\nu(B_k) < \infty$ for each k and $\cup_k A_k = X$ and $\cup_k B_k = Y$, and let $f_k(x) = \min\{f, k\} \, \chi_{A_k \times B_k}$. Then f_k is an increasing sequence of γ-integrable functions with $\lim f_k = f$, and so Fubini's theorem gives

$$\int_{X\times Y} f_k \, d\gamma = \int_Y \Big(\int_X f_k(x,y)\, d\mu(x)\Big) d\nu(y),$$

and by applying the monotone convergence theorem (once on the left and twice on the right) we deduce the required result.