Mathematics Department, Stanford University
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Lebesgue's theorem on the Riemann integral

We let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be any closed interval in $\mathbb{R}^n$. Recall that by a *partition* $\mathcal{P}$ of R we mean the collection of closed intervals $I \subset R$ obtained by partitioning each of the edges of R; thus for each $j = 1, \ldots, n$ we select points $a_j = t_{j,0} < t_{j,1} < \cdots < t_{j,N_j} = b_j$ and then $\mathcal{P} = \{[t_{1,i_1-1}, t_{1,i_1}] \times [t_{2,i_2-1}, t_{2,i_2}] \times \cdots \times [t_{n,i_n-1}, t_{n,i_n}] : i_j \in \{1, \ldots, N_j\}$ for each $j = 1, \ldots, n\}$. The points $t_{j,0}, \ldots, t_{j,N_j}$ are called "the j-th edge points" of the partition $\mathcal{P}$. For any $I = [t_{1,i_1-1}, t_{1,i_1}] \times [t_{2,i_2-1}, t_{2,i_2}] \times \cdots \times [t_{n,i_n-1}, t_{n,i_n}] \in \mathcal{P}$ we let $\breve{I}$ denote the corresponding open interval $(t_{1,i_1-1}, t_{1,i_1}) \times (t_{2,i_2-1}, t_{2,i_2}) \times \cdots \times (t_{n,i_n-1}, t_{n,i_n})$, and $\partial I = I \setminus \breve{I}$.

Corresponding to any such partition $\mathcal{P}$ of R, $U(f, \mathcal{P}) = \sum_{I \in \mathcal{P}} (\sup_I f)\, |I|$ is the "upper Riemann sum" and $L(f, \mathcal{P}) = \sum_{I \in \mathcal{P}} (\inf_I f)\, |I|$ is the "lower Riemann sum," where $|I|$ is the volume of I (i.e. the product of the edge lengths of I), and recall the a bounded function $f : R \to \mathbb{R}$ is Riemann integrable if $\underline{\int}_R f = \overline{\int}_R f$, where

$$\underline{\int}_R f = \sup_{\text{partitions } \mathcal{P} \text{ of } R} L(f, \mathcal{P}), \ \overline{\int}_R f = \inf_{\text{partitions } \mathcal{P} \text{ of } R} U(f, \mathcal{P}).$$

Recall also that then we have the "Riemann criterion," which says that f is Riemann integrable on R if and only if for each $\delta > 0$ there is a partition $\mathcal{P}$ of R such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \delta$.

Theorem. *Let $f : [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbf{R}$ be a bounded function. f is Riemann integrable $\iff$ there is a set $A \subset [a_1, b_1] \times \cdots \times [a_n, b_n]$ of Lebesgue measure zero such that f is continuous at each point of $[a_1, b_1] \times \cdots \times [a_n, b_n] \setminus A$.*

(i.e. A bounded function f on an interval $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is Riemann integrable if and only if f is continuous a.e. in R.)

Cautionary Remark: "$f : R \to \mathbb{R}$ is continuous at each $x \in R \setminus A$" is a much stronger condition than "$f|R \setminus A$ is a continuous function," and indeed $f|R \setminus A$ continuous is in general not sufficient to ensure that f is Riemann integrable even if A has measure zero. For example if we take $R = [0, 1]$, $A =$ the set of rationals in $[0, 1]$, then A has measure zero but the function f which is 1 on A and 0 on $R \setminus A$ is not Riemann integrable because evidently $\underline{\int}_R f = 0$ and $\overline{\int}_R f = 1$.

Proof of $\Rightarrow$: Observe, by the definition of continuity, that f discontinuous at $y \in (a_1, b_1) \times \cdots \times (a_n, b_n) \iff \exists\, \varepsilon_0 > 0$ such that $\sup_I f - \inf_I f > \varepsilon_0$ $\forall$ open interval I with $y \in I \subset (a_1, b_1) \times \cdots \times (a_n, b_n)$, which is the same as saying there is a positive integer j such that $\sup_I f - \inf_I f > 1/j$ $\forall$ open interval I with $y \in I \subset (a_1, b_1) \times \cdots \times (a_n, b_n)$. Thus the set of discontinuities of $f|(a_1, b_1) \times \cdots \times (a_n, b_n)$ can be written $\cup_{j=1}^{\infty} S_j$, where

$$S_j = \{y \in (a_1, b_1) \times \cdots \times (a_n, b_n) : \sup_I f - \inf_I f > 1/j \quad \text{for every open interval } I \text{ with } y \in I \subset (a_1, b_1) \times \cdots \times (a_n, b_n)\}.$$

Since the countable union of sets of Lebesgue measure zero again has Lebesgue measure zero, it is thus enough to prove that S_j has Lebesgue measure zero for each j.

Let $\varepsilon > 0$, $j \in \{1, 2, \ldots\}$, and note that by the above Riemann criterion we can pick a partition $\mathcal{P}$ of the R such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon/j$. That is,

$$\sum_{I\in\mathcal{P}} \Big(\sup_I f - \inf_I f\Big)|I| < \varepsilon/j.$$

Since $\sup_{I\in\mathcal{P}} f - \inf_I f \geq \sup_{\breve{I}} f - \inf_{\breve{I}} f \geq 1/j$ whenever $S_j \cap \breve{I} \neq \emptyset$ (by definition of S_j), where $\breve{I}$ denotes the open interval $I \setminus \partial I$, the above evidently implies

$$\sum_{\{i\,:\, S_j\cap\breve{I}\neq\emptyset\}} (1/j)|I| < \varepsilon/j;$$

that is,

$$(\ddagger) \qquad \sum_{\{I\in\mathcal{P}: S_j\cap\breve{I}\neq\emptyset\}} |I| < \varepsilon.$$

But the intervals I, $I \in \mathcal{P}$, cover the entire interval R, hence $R \setminus \cup_{I\in\mathcal{P}} \partial I = \cup_{I\in\mathcal{P}} \breve{I}$ and trivially therefore $S_j \setminus \cup_{I\in\mathcal{P}} \partial I \subset \cup_{\{I\in\mathcal{P}:S_j\cap\breve{I}\neq\emptyset\}} \breve{I}$. Of course ∂I has Lebesgue measure zero for each $I \in \mathcal{P}$, so ($\ddagger$) proves that S_j can be covered by a finite union of intervals of total length $< \varepsilon$ and hence S_j has Lebesgue measure zero as required.

Proof of $\Leftarrow$: Let $\varepsilon > 0$ and cover the set S of discontinuities of f by a countable union I_j of open intervals such that $\sum_j |I_j| < \varepsilon$. Then $K \equiv R \setminus \cup_{j=1}^{\infty} I_j$ is a compact set and f (as a function of $x \in R$) is continuous at each point of this compact set. We can therefore assert that

$$(*) \qquad \exists\, \delta > 0 \text{ such that } |f(x) - f(y)| < \varepsilon \text{ whenever } x \in K,\ y \in R, \text{ and } |x - y| < \delta.$$

Notice that the statement $(*)$ is stronger than the standard fact that a continuous function on a compact set is uniformly continuous, because only the point x, and not necessarily the point y, is required to be in the compact set K—on the other hand, the proof using the Bolzano-Weierstrass theorem is almost identical to the usual Bolzano-Weierstrass proof of this standard fact, as follows: If there is $\varepsilon > 0$ such that $(*)$ fails for each $\delta > 0$ then it fails with $\delta = \frac{1}{k}$, $k = 1, 2, \ldots$, and hence there are points $x_k \in K, y_k \in R$ such that $|x_k - y_k| < \frac{1}{k}$ but $|f(x_k) - f(y_k)| \geq \varepsilon$. Then by the Bolzano-Weierstrass theorem we can find a convergent subsequence x_{k_j} with $x = \lim x_{k_j}$, and $x \in K$ because K is closed. Since $|x_{k_j} - y_{k_j}| < \frac{1}{k_j} \leq \frac{1}{j}$ we also have $\lim y_{k_j} = x$, and so by continuity of f at x we have $f(x_{k_j}) - f(y_{k_j}) \to f(x) - f(x) = 0$, contradicting the fact that $|f(x_{k_j}) - f(y_{k_j})| \geq \varepsilon$ for each j.

Now, with such a δ, we select any partition $\mathcal{P}$ of R with $\operatorname{diam} I < \delta$ for each $I \in \mathcal{P}$. For any $I \in \mathcal{P}$ such that $I \cap K \neq \emptyset$ we have by $(*)$ that

$$\begin{aligned}\sup_I f - \inf_I f &= \sup_{z_1,z_2\in I}(f(z_1) - f(z_2)) \\ &= \sup_{z_1,z_2\in I}((f(z_1) - f(y_I)) - (f(z_2) - f(y_I))) \leq \varepsilon + \varepsilon = 2\varepsilon,\end{aligned}$$

where y_I is any point in $I \cap K$, while of course the sum of the volumes $|I|$ over the remaining $I \in \mathcal{P}$ is $\leq \varepsilon$ (because these remaining intervals I have the property $I \cap K = \emptyset$ and hence $I \subset R \setminus K = R \setminus (R \setminus (\cup_j I_j)) \subset \cup_j I_j$). Thus we have

$$\begin{aligned}U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{I\in\mathcal{P}} (\sup_I f - \inf_I f)\,|I| \\ &\leq 2\varepsilon|R| + (\sup_R f - \inf_R f)\varepsilon \leq 2\varepsilon(|R| + M), \quad M = \sup_R |f|.\end{aligned}$$